Examples of moduli spaces of sheaves on K3 surfaces

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1 Introduction

In this talk, X will always be a smooth projective variety over an algebraically closed field k, $\operatorname{char} k = 0$, equipped with a polarization $\mathcal{O}_X(1)$. (Since we'll only be working with examples today, there's no harm in supposing that $k = \mathbb{C}$.) We will soon specialize to K3 surfaces, and will mostly only be concerned with rank-2 (semi)stable sheaves.

Last week, we finished the construction of the moduli space $M^{(s)}_{\mathcal{O}_X(1)}(P)$ of semistable (stable) sheaves on X with fixed Hilbert polynomial P. Today we'll forget the general construction, and move on to Huybrechts and Lehn's main goal, studying the geometry of these moduli spaces. The moral of the story will be that the moduli spaces have an uncanny tendency to inherit the geometric properties of the spaces they arise from. For example, although we will only discuss two specific examples today, it holds in much greater generality that the triviality of K_X implies the triviality of K_M.

2 Preliminary facts

Fix a Hilbert polynomial P for the moment, so that we can discuss the moduli space $M(P) = M_{\mathcal{O}_X(1)}(P)$. By "taking determinant bundles of sheaves", we get a map $M(P) \to \operatorname{Pic}(X)$. What this really means is that $M(P)$ corepresents the functor $\mathcal{M}_{\mathcal{O}_X(1)}(P)$, and this functor admits a natural transformation to the (representable) Picard functor, so this natural transformation must factor through a morphism $M(P) \to \operatorname{Pic}(X)$ of schemes. This allows us to fix another invariant of our sheaves. In particular, if Q is a line bundle on X, $[Q]$ is a closed point of $\operatorname{Pic}(X)$, so we can let $M(P,Q)$ denote the fiber of $M(P)$ over $[Q]$. This parametrizes (S-equivalence classes of) semistable sheaves with determinant bundle Q. Define $M^s(P,Q)$ similarly.

Using the obstruction theory ideas that Sherman mentioned last week, one can prove upper and lower bounds for $\dim_{[F]} M(P,Q)$, if F is stable:

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Theorem 2.1. *Let F be a stable $\mathcal{O}_X$-module of rank $r > 0$ and determinant bundle Q, and let $M(P,Q)$ be as above. Then $T_{[F]}M(P,Q) \cong \mathrm{Ext}^1(F,F)_0$, and*

$$\mathrm{ext}^1(F,F)_0 - \mathrm{ext}^2(F,F)_0 \leq \dim_{[F]} M(P,Q) \leq \mathrm{ext}^1(F,F)_0. \tag{1}$$

If $\mathrm{Ext}^2(F,F)_0 = 0$*, then $M(P)$ and $M(P,Q)$ are smooth at $[F]$.*

(Here, $\mathrm{Ext}^i(F,F)_0$ is the kernel of a natural map $\mathrm{Ext}^i(F,F) \to H^i(\mathcal{O}_X)$, which is always surjective in characteristic 0.)

Definition 2.1. *We call the lower bound* $\mathrm{ext}^1(F,F)_0 - \mathrm{ext}^2(F,F)$ *the* expected dimension *of $M(P,Q)$ at F.*

This expected dimension turns out to equal $(r^2-1)(g-1)$ for curves—in which case the vanishing of $\mathrm{Ext}^2(F,F)_0$ is vacuous—and $\Delta(F) - (r^2-1)\cdot\chi(\mathcal{O}_X)$ for surfaces. (In the case of surfaces, it can be shown that the error term has a bound depending only on X and r.)

To explain the notation above: $\Delta(F)$ is the "discriminant" $\Delta(F) = 2rc_2(F) - (r-1)c_1^2(F)$, viewed as an integer via the degree map $A^n(X) \to \mathbb{Z}$; the c_i are Chern classes.

Since we'll be talking more about Chern classes, let me take a moment to recall some basic facts about them that we'll use. Given a vector bundle (or coherent sheaf?) F on X, the Chern classes $c_i(F) \in A^i(X)$ are a sequence of naturally defined elements of the Chow ring. (You may have also heard of Chern classes in $H^{2i}_{sing}(X^{an},\mathbb{Z})$, which receives a natural map from $A^i(X)$.) They vanish when $i > \operatorname{rank} F$, and of course also when $i > \dim X$. For any product of Chern classes that lies in $A^n(X)$, we get a numerical invariant of F via the degree map $A^n(X) \to \mathbb{Z}$. For example, if X is a surface, we have two independent Chern numbers, c_2 and c_1^2. Finally, $c_1(F)$ has a concrete interpretation: when $F = \mathcal{O}(D)$ is a line bundle, $c_1(F)$ is just $[D] \in A^1(X)$. In general, $c_1(F) = c_1(\det F)$.

3 K3 surfaces

A K3 surface X/k is defined to be a smooth, proper, geometrically irreducible surface with trivial canonical bundle $K_X \cong \mathcal{O}_X$, and with $H^1(X,\mathcal{O}_X) = 0$.

Geometrically speaking, the first condition tells us that there is a unique globally-defined algebraic volume form on X. This is something like saying that X has curvature 0 as a complex manifold. Drawing a comparison to the trichotomy of curves, varieties with this property (called Calabi-Yau) are analogous to elliptic curves. The second condition excludes the other obvious surface analog of elliptic curves, namely abelian surfaces.

K3 surfaces can be seen as analogs of elliptic curves in another sense: they are often at the bounds of what is complicated but tractable. This is true arithmetically as well as geometrically: for example, K3 surfaces are among the few classes of varieties for which the Tate conjecture is known. **(Check on this.)**

Just to be able to get our hands on some K3 surfaces, let's describe those K3's that are complete intersections in $\mathbb{P}^N$. Suppose that X is a complete intersection in $\mathbb{P}^N$ of smooth hypersurfaces of degrees $d_1, \ldots, d_{N-2}$. It can be checked that $K_X = \mathcal{O}_{\mathbb{P}^N}(-N-1+\sum d_i)|_X$, so we need $\sum d_i = N+1$ in order for X to be K3. All such complete intersections turn out to actually be K3's. Notice, moreover, that if we assume without loss of generality that all $d_i > 1$, then we get very few options for $(N, \{d_i\})$: X can be a quartic in $\mathbb{P}^3$, the intersection of a cubic and a quadric in $\mathbb{P}^4$, or the intersection of three quadrics in $\mathbb{P}^5$.

Before starting our first example, let's write down a quick lemma:

Lemma 3.1. *If X is a K3 surface, then all moduli spaces $M^s(2, \det = Q, \Delta)$ are either empty or smooth of expected dimension $\Delta - (r^2-1)\chi(X, \mathcal{O}_X) = \Delta - 6 = 4c_2 - c_1^2 - 6$.*

Notational aside: up until now, we have been considering moduli spaces of sheaves with a given Hilbert polynomial. It may appear that the Hilbert polynomial is absent from the new notation that Huybrechts and Lehn have sprung upon us (without explanation!), but it is in fact there. The Hilbert polynomial contains exactly the same information as the rank and Chern classes. (This can be seen from Riemann-Roch for surfaces, or more generally Grothendieck-Riemann-Roch.) It may still appear that the first Chern class is missing, but we can recover it as $c_1(\det F)$. I don't have a nice, succinct formula expressing the correspondence between the two types of data—I think it relies on knowledge of the canonical divisor, the Euler characteristic of X, the polarization, and various intersection numbers—but it is true that these are just two different ways to package the same information.

Proof. Fix a stable sheaf F; we want to show that the obstruction space $\mathrm{Ext}^2(E,E)_0$ vanishes. Recall that for X a K3 surface, we have $\omega_X = K_X = \mathcal{O}_X$. By a version of Serre duality, we have $\mathrm{Ext}^i(A,B) = \mathrm{Ext}^{2-i}(B, A \otimes \omega_X)^\vee = \mathrm{Ext}^{2-i}(B,A)^\vee$ for A, B any coherent sheaves. We have seen that $\mathrm{Hom}(E,E) = k$, so $\mathrm{Ext}^2(E,E) = \mathrm{Ext}^0(E,E)^\vee \cong k$. But $\mathrm{Ext}^2(E,E)_0$ is by definition the kernel of a surjective map $\mathrm{Ext}^2(E,E) \to H^2(X, \mathcal{O}_X)$, so it must vanish. It follows by our earlier criterion that $M(P,Q) = M(2,Q,\Delta)$ is smooth of the expected dimension at all stable points; i.e. $M^s(2,Q,\Delta)$ is either empty or smooth of expected dimension. Finally, to actually compute the expected dimension, recall that $\chi(X, \mathcal{O}_X) = 1 - 0 + 1 = 2$. □

4 First example

Let X be a "general" K3 surface of the first type above, namely a smooth quartic surface in $\mathbb{P}^3$. The Noether-Lefschetz theorem asserts that $\mathrm{Pic}(X)$ is generated by $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^3}(1)|_X$ for a general such surface, so we have a canonical choice of polarization. We are interested in the moduli spaces $M(r=2, \det = \mathcal{O}_X(-1), c_2)$, for c_2 an integer. **Fact (why?): all μ-semistable sheaves with this determinant are μ-stable. This doesn't seem to come from Corollary 5.3.3, because I don't think we generally have a fibration.** So $M(2, \mathcal{O}_X(-1), c_2) = M^s(2, \mathcal{O}_X(-1), c_2)$. By our lemma, this is either empty or smooth of dimension $4c_2 - c_1^2 - 6$. But c_1 can be determined in terms of $\det = \mathcal{O}_X(-1)$: whenever $\det F = \mathcal{O}_X(-1)$, we have $c_1(F) = -[H] \in A^1(X)$, where H is a hyperplane divisor so that $\mathcal{O}_X(-1) = \mathcal{O}_X(-H)$. Then $c_1^2 = (-H).(-H)$, the self-intersection number of the hyperplane class, which is $\deg X = 4$. So $M(2, \mathcal{O}_X(-1), c_2)$ is either empty or smooth of dimension $4c_2 - 10$.

What c_2 are we interested in? If $c_2 \leq 2$, then the expected dimension above is negative, which proves that the moduli spaces are empty. If $c_2 = 3$, we expect a surface; if 4, a 6-dimensional smooth variety, and so on. Let's consider the first nontrivial case, $c_2 = 3$. We hope that this moduli space will be nonempty, and thus a smooth projective surface. Moreover, based on our philosophy that geometric properties of X carry over to the moduli spaces of its sheaves, we hope that this surface will somehow resemble X. We might even dare to hope that:

Claim 4.1. *The moduli space $M(r = 2, \det = \mathcal{O}_X(-1), c_2 = 3)$ is isomorphic to X.*

Proof. **(Sketch.)** □

5 Second example

Let X be a K3 surface, and suppose X admits an elliptic fibration $\pi : X \to \mathbb{P}^1$ (i.e. the general fiber is a smooth curve of genus 1), with irreducible fibers. Suppose also that π admits a section $\sigma \subset X$. It follows from the adjunction formula that σ is a (-2)-curve. Now let f be the divisor given by a fiber of π. Let $H = \sigma + 3f$, and let $H_m = H + mf$ for $m \geq 0$. By the Nakai-Moishezon criterion, these H_m are ample divisors.

Consider the moduli spaces $M_{H_m}(r = 2, \det, c_2)$. If we fix c_1 and c_2 with $c_1.f$ odd, and take m sufficiently large, we can again brush aside the issue of semistability versus stability, thanks to the technical criterion of Corollary 5.3.3.

Claim 5.1. *If m is sufficiently large, then $M = M_{H_m}(r = 2, \det = \mathcal{O}_X(\sigma - f), c_2 = 1) \cong X$.*

To clarify some things in the statement: the choice of polarization actually matters here, and we are choosing H_m. It is true that $c_1.f$ is odd, because $c_1 = \sigma - f$, and we have $\sigma.f = 1$ and $f.f = 0$. Another sanity check: the expected dimension of this moduli space is $\Delta - 6$, where

$$\begin{aligned} \Delta &= 4c_2 - c_1^2 && (2)\\ &= 4 - (\sigma - f)^2 && (3)\\ &= 4 - (\sigma^2 - 2\sigma.f + f^2) && (4)\\ &= 4 - (-2 - 2 + 0) = 8. && (5) \end{aligned}$$

So we do actually expect a surface.

Proof. Huybrechts and Lehn give two proofs, one using the Serre correspondence and one using elementary transformations. (We certainly do not need to work through the general moduli space construction!) Instead of giving these proofs in detail, let's just briefly state the fundamental idea of each of these methods.

The Serre correspondence is a process by which we can associate to a codimension-two subvariety (i.e. a closed point) a rank-2 vector bundle. More precisely, for $Z \subset X$ a local complete intersection of codimension 2, $\mathcal{I}_Z$ its ideal sheaf, and L and M any line bundles on X, the Serre correspondence gives a necessary and sufficient condition for the existence of a vector bundle E fitting into a short exact sequence

$$0 \to L \to E \to M \otimes \mathcal{I}_Z \to 0. \tag{6}$$

In this case, one can prove the claim by showing (more or less) that every stable rank-2 bundle on X arises uniquely from the Serre correspondence.

Elementary transformations give another method of constructing vector bundles on X. If $i : C \hookrightarrow X$ is an effective divisor on X, F is a vector bundle on X, and G is a vector bundle on C, we say E is an elementary transformation of F along G if it fits into a short exact sequence

$$0 \to E \to F \to i_* G \to 0. \tag{7}$$

It can be shown that every rank-r vector bundle on X can be written as an elementary transformation of $\mathcal{O}_X^{\oplus r}(nH)$ along a suitable line bundle on a curve on X, for sufficiently large n. $\square$